Analysis on ultra-metric spaces and heat kernels

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Heat kernels in $\mathbb{R}^n$

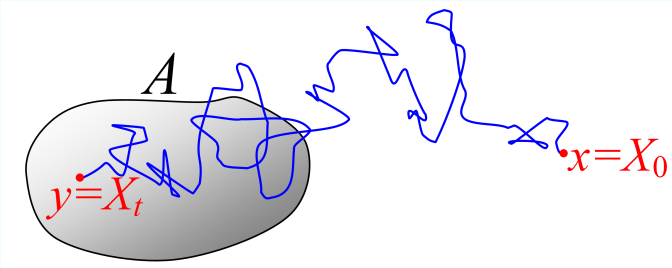
The heat equation $\frac{\partial u}{\partial t} = \Delta u$ in $\mathbb{R}^n$ has the following *heat kernel*

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{1}$$

It coincides with the transition density of Brownian motion $\{X_t\}_{t\geq 0}$ in $\mathbb{R}^n$:

$$\mathbb{P}_x\left(X_t \in A\right) = \int_A p_t(x,y)dy$$

for any Borel set $A \subset \mathbb{R}^n$.



The operator $-\Delta$ can be extended to a self-adjoint non-negative definite operator in $L^2(\mathbb{R}^n)$, which allows to define the *heat semigroup* $\left\{e^{t\Delta}\right\}_{t\geq 0}$. The operator $e^{t\Delta}$ for $t>0$ is an integral operator with the integral kernel $p_t(x,y)$.

For any $\beta \in (0,2)$ the operator $(-\Delta)^{\beta/2}$ is also a self-adjoint non-negative definite operator, and the associated the heat semigroup $\{e^{-t(-\Delta)^{\beta/2}}\}_{t\geq 0}$ has a non-negative integral kernel $p_t^{(\beta)}(x,y)$ that coincides with the transition density of a *symmetric stable Levy process* of index β in $\mathbb{R}^n$.

It is known that in the case $\beta = 1$

$$p_t^{(1)}(x,y) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}} \tag{2}$$

that is, $p_t^{(1)}(x,y)$ is the density of the Cauchy distribution with the parameter t.

For any $\beta \in (0,2)$, the heat kernel of $(-\Delta)^{\beta/2}$ satisfies the estimate

$$p_t^{(\beta)}(x,y) \simeq \frac{t}{\left(t^{1/\beta} + |x-y|\right)^{n+\beta}} = \frac{1}{t^{n/\beta}} \left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}. \tag{3}$$

The sign $\simeq$ means that the ratio of two sides is bounded between two positive constants. The formulas (2) and (3) are obtained from the heat kernel (1) of Δ by using subordination techniques (=integral transform of the heat kernel of Δ).

Dirichlet forms of jump type

Let (M,d) be a locally compact separable metric space and μ be a Radon measure on M with full support. The theory of Dirichlet forms of M.Fukushima provides the following method of construction of Markov jump processes on M. Consider in $L^2(M,\mu)$ the following quadratic form

$$\mathcal{E}(f,f)=\iint_{M\times M}(f(x)-f(y))^2 J(x,y)d\mu(x)d\mu(y), \tag{4}$$

where $J(x,y)$ is a non-negative symmetric function on $M\times M$. Assume that $\mathcal{E}$ extends to a regular Dirichlet form with a domain $\mathcal{F}\subset L^2(M,\mu)$, i.e. $\mathcal{F}$ is a dense subspace of L^2 and has "a lot" of continuous functions. Its generator

$$\mathcal{L}f(x)=\int_M (f(y)-f(x))\,J(x,y)d\mu(y)$$

is a self-adjoint non-positive definite operator in L^2. The *heat kernel* $p_t(x,y)$ of $(\mathcal{E},\mathcal{F})$ is the integral density of the heat semigroup: for all $t>0$ and $f\in L^2$

$$e^{t\mathcal{L}}f(x)=\int_M p_t(x,y)f(y)d\mu(y).$$

Besides, there exists a Markov process with the transition density $p_t(x,y)$.

For example, consider $\mathbb{R}^n$ the quadratic form (4) with $J(x,y) = |x-y|^{-(n+\beta)}$:

$$\mathcal{E}(f,f) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x)-f(y))^2}{|x-y|^{n+\beta}} dx\, dy. \tag{5}$$

If $0 < \beta < 2$ then (5) extends to a regular Dirichlet form with the generator $-(-\Delta)^{\beta/2}$ and heat kernel (3).

Question: Under what conditions on a metric measure space (M, d, μ) and a jump kernel J, the heat kernel of the associated Dirichlet form exists and satisfies for all $x, y \in M$ and $t > 0$ a *stable-like estimate*

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{6}$$

with some positive parameters α, β? It is motivated by the following theorem.

Theorem 1 (AG and T.Kumagai 2008). *Assume that the heat kernel of a conservative jump type Dirichlet form satisfies the estimate*

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

for some function Φ and for all $x, y \in M$ and $t > 0$. Then it has to be (6).

The following *necessary* conditions for (6) are known:

- the α-regularity: for any metric ball $B_r(x)$, we have

$$\mu\left(B_r(x)\right) \simeq r^\alpha \tag{V}$$

(consequently, $\alpha = \dim_H M$ and $\mu \simeq \mathcal{H}_\alpha$).

- the jump kernel estimate: for all $x, y \in M$,

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

In general, (V) and (J) are not enough for (6) – one needs in addition a *generalized capacity condition* (AG, E.Hu, J.Hu, *Adv. Math.* 2018 and Z.Q.Chen, T.Kumagai, J.Wang, to appear in *Memoirs AMS*).

However, if (M,d) is an *ultra-metric* space then the situation is better.

Ultra-metric spaces

Let (M,d) be a metric space. The metric d is called an *ultra-metric* and (M,d) is called an *ultra-metric space* if, for all $x,y,z \in M$,

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}. \tag{7}$$

Define the metric balls

$$B_r(x) = \{y \in M : d(x, y) \leq r\}.$$

The ultra-metric property (7) implies that *any two metric balls of the same radius are either disjoint or identical.* Another consequence: *every point inside a ball is its center.* Therefore, all balls are closed and open sets, and M is totally disconnected. In particular, an ultra-metric space cannot carry a non-trivial diffusion process.

A well known example of an ultra-metric space is the set $\mathbb{Q}_p$ of p-adic numbers with the ultra-metric $\|x-y\|_p$. Consider also the space $\mathbb{Q}_p^n$ with the p-adic norm

$$\|x\|_p := \max_{1\leq i\leq n} \|x_i\|_p\,, \quad x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n,$$

and the ultra-metric $d(x, y) = \|x-y\|_p$. Let μ be the Haar measure on $\mathbb{Q}_p^n$ with the normalization condition

$$\mu(B_1(x)) = 1.$$

Then we have for any $m \in \mathbb{Z}$

$$\mu(B_{p^m}(x)) = p^{nm}.$$

If $p^m \leq r < p^{m+1}$ then $B_r(x) = B_{p^m}(x)$ which implies

$$\mu(B_r(x)) = p^{nm} \simeq r^n. \tag{8}$$

Taibleson operator in $\mathbb{Q}_p^n$

Fourier transform of a function $f : \mathbb{Q}_p^n \to \mathbb{C}$ is a function $\widehat{f} : \mathbb{Q}_p^n \to \mathbb{C}$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p^n} e^{2\pi i \langle x,\xi \rangle} f(x) d\mu(x),$$

where $\xi = (\xi_1, ..., \xi_n) \in \mathbb{Q}_p^n$, μ is the Haar measure on $\mathbb{Q}_p^n$, and

$$\langle x, \xi \rangle = \sum_{k=1}^{n} \{x_k \xi_k\},$$

where $\{z\}$ denotes the fractional part of $z \in \mathbb{Q}_p$ and, hence, $\{z\} \in \mathbb{Q}$.
Define the operator $\mathcal{T}^\beta$ for any $\beta > 0$ by means of its Fourier transform:

$$\widehat{\mathcal{T}^\beta f}(\xi) = \|\xi\|_p^\beta \, \widehat{f}(\xi). \tag{9}$$

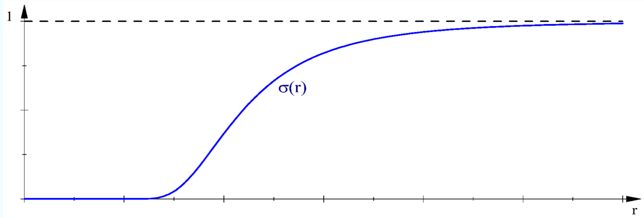
This operator was introduced by Taibleson in 1975 as an analogue of $(-\Delta)^{\beta/2}$ in $\mathbb{R}^n$. The operator $\mathcal{T}^\beta$ is self-adjoint and non-negative definite in L^2, and

$$T^\beta = \left(T^1\right)^\beta.$$

Isotropic Dirichlet forms

Let (M, d) be an ultra-metric space where all balls are compact. Let μ be a Radon measure on M with full support.

Let $\sigma(r)$ be a cumulative probability distribution function on $(0, \infty)$ that is strictly monotone increasing.



Consider on M a jump kernel

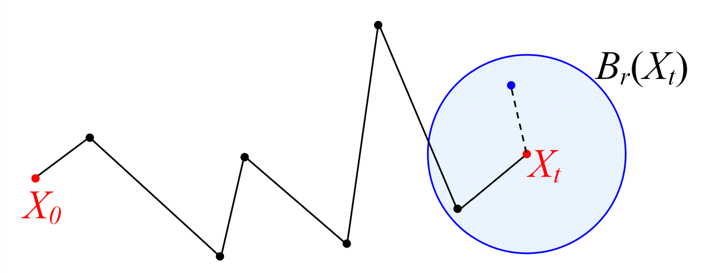
$$J(x, y) = \int_{d(x,y)}^{\infty} \frac{d \log \sigma(r)}{\mu(B_r(x))}. \tag{10}$$

Theorem 2 (A.Bendikov, AG, Ch.Pittet, W.Woess, Uspechi, 2014) *The jump kernel* (10) *determines a regular Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *in* $L^2(M, \mu)$, *and its heat kernel is*

$$p_t(x, y) = \int_{d(x,y)}^{\infty} \frac{d\sigma^t(r)}{\mu(B_r(x))}. \tag{11}$$

The Dirichlet form with the jump kernel (10) is called an *isotropic Dirichlet form.*

The jump process $\{X_t\}_{t\geq 0}$ generated by $(\mathcal{E}, \mathcal{F})$ looks as follows:



At any t, it jumps from X_t to the next position by uniformly distributing in $B_r(X_t)$, where r is randomly chosen by using the probability distribution σ. The ultra-metric property is used in the proof as follows. On an ultra-metric space, the averaging operators

$$Q_r f(x) = \fint_{B_r(x)} f(y) d\mu(y)$$

are bounded in $L^2(M, \mu)$, self-adjoint, and satisfy the identity

$$Q_r Q_s = Q_s Q_r = Q_{\max\{r,s\}} \quad \text{for all } r, s > 0. \tag{12}$$

In particular, $Q_r^2 = Q_r$, that is, Q_r is an orthoprojector in L^2.

Indeed, for any ball B of radius r, *any* point $x \in B$ is a center of B. Since the value $Q_r f(x)$ is the average of f in B, we see that $Q_r f(x)$ does not depend on $x \in B$. Hence, $Q_r f = \text{const}$ on any ball of radius r.

If $s \leq r$ then the application of Q_s to $Q_r f$ does not change this constant, whence we obtain $Q_s Q_r f = Q_r f$.

If $s > r$ then any ball of radius s is the disjoint union of finitely many balls of radius r. Since the integrals of f and $Q_r f$ over any such ball are the same, we obtain $Q_s Q_r f = Q_s f$.

The property (12) is used to prove that the family of operators

$$P_t = \int_0^\infty Q_r \, d\sigma^t(r), \quad t > 0,$$

is a semigroup and that it coincides with the heat semigroup $e^{t\mathcal{L}}$ of the isotropic Dirichlet form, which leads to (11).

Let us mention for comparison, that the averaging operator Q_r in $\mathbb{R}^n$ is also bounded and self-adjoint, but it has a non-empty negative part of the spectrum. In particular, it is *not* an orthoprojector.

For example, consider $M = \mathbb{Q}_p^n$ with the ultra-metric

$$d(x,y) = \max_{1\leq i\leq n} \|x_i - y_i\|_p$$

and with the Haar measure μ as above. Fix any $\beta > 0$ and consider the distribution function

$$\sigma(r) = \exp\left(-\left(\frac{p}{r}\right)^\beta\right) \tag{13}$$

(Fréchet distribution). Substituting (13) into (10) and using the exact formula (8) for $\mu(B_r(x))$, we obtain

$$J(x,y) = c_{p,n,\beta}\, d(x,y)^{-(n+\beta)}, \tag{14}$$

which miraculously coincides with the jump kernel of the Taibleson operator $\mathcal{T}^\beta$! (see (9)). Hence, the generator $\mathcal{L}$ of the isotropic Dirichlet form with this σ coincides with $\mathcal{T}^\beta$. Substituting (13) into (11), we obtain that the heat kernel of T^β satisfies

$$p_t(x,y) \simeq \frac{1}{t^{n/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(n+\beta)},$$

that is, the stable like estimate (6) with $\alpha = n$. Let us emphasize that $\mathcal{T}^\beta$ generates a Markov jump process for any $\beta > 0$ unlike the case of $\mathbb{R}^n$ where $(-\Delta)^{\beta/2}$ generates such a process only if $\beta \in (0,2)$.

Jump kernels on α-regular ultra-metric spaces

Let an ultra-metric space (M, d, μ) satisfy (V) for some $\alpha > 0$: $\mu(B_r(x)) \simeq r^\alpha$ for all $x \in M$ and $r > 0$. Consider a jump kernel on M such that, for some $\beta > 0$,

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

The associated Dirichlet form is not necessarily isotropic, and the above method does not work. The results below were proved by A.Bendikov, AG, E. Hu, J.Hu, *Ann. Scuola Norm. Sup. Pisa, 2021.*

Theorem 3 *Assume that* (V) *and* (J) *are satisfied. Then the quadratic form*

$$\mathcal{E}(f,f) = \iint_{M\times M} (f(x) - f(y))^2 J(x,y)d\mu(x)d\mu(y)$$

determines a regular Dirichlet form in $L^2(M,\mu)$*. Its heat kernel* $p_t(x,y)$ *exists, is continuous in* (t,x,y)*, Hölder continuous in* (x,y) *and satisfies the stable-like estimate* (6)*, that is, for all* $x, y \in M$ *and* $t > 0$*,*

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{15}$$

Consequently, (V)+$(J)$$\Leftrightarrow$(15).

Product spaces

Consider a sequence $\{(M_i, d_i, \mu_i)\}_{i=1}^n$ of ultra-metric measure spaces such that M_i is α_i-regular. Consider on M_i the jump kernel

$$J_i(x, y) = d_i(x, y)^{-(\alpha_i+\beta)}$$

where $\beta > 0$ is the same for all i. By Theorem 3, the heat kernel $p_t^{(i)}(x, y)$ on M_i satisfies the estimate

$$p_t^{(i)}(x, y) \simeq \frac{1}{t^{\alpha_i/\beta}} \left(1 + \frac{d_i(x, y)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)}.$$

Consider the product space $M = M_1 \times ... \times M_n$ with the ultra-metric

$$d(x, y) = \max_{1 \le i \le n} d_i(x_i, y_i)$$

and with the product measure $\mu = \mu_1 \times ... \times \mu_n$. Then M is α-regular with $\alpha = \alpha_1 + ... + \alpha_n$. Consider on M the operator

$$\mathcal{L} = \mathcal{L}_1 + ... + \mathcal{L}_n$$

where $\mathcal{L}_i$ in the generator in M_i acting on the coordinate x_i.

Then $\mathcal{L}$ generates a Dirichlet form on M with the jump measure (not kernel!):

$$J(x,dy)=\sum_{i=1}^{n}\delta_{x_1}(y_1)\times\ldots\times\delta_{x_{i-1}}(y_{i-1})\times J_i(x_i,y_i)d\mu_i(y_i)\times\delta_{x_{i+1}}(y_{i+1})\ldots\times\delta_{x_n}(y_n). \tag{16}$$

The heat kernel of $\mathcal{L}$ is given by

$$p_t(x,y)=\prod_{i=1}^{n}p_t^{(i)}(x_i,y_i)\simeq\frac{1}{t^{\alpha/\beta}}\prod_{i=1}^{n}\left(1+\frac{d_i(x_i,y_i)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)}. \tag{17}$$

For example, consider the *Vladimirov operator* $\mathcal{V}^\beta$ in $\mathbb{Q}_p^n$ defined by

$$\mathcal{V}^\beta=\sum_{i=1}^{n}\mathcal{T}_{x_i}^{\beta},$$

where $\mathcal{T}_{x_i}^{\beta}$ is the Taibleson operator in $\mathbb{Q}_p$ acting on the coordinate x_i. In this case $\alpha_i=1$, $\alpha=n$, and we obtain that the kernel of $\mathcal{V}^\beta$ satisfies

$$p_t(x,y)\simeq\frac{1}{t^{n/\beta}}\prod_{i=1}^{n}\left(1+\frac{\|x_i-y_i\|_p}{t^{1/\beta}}\right)^{-(1+\beta)}.$$

Walk dimension

We say that a metric space (M,d) is regular if it admits an α-regular measure μ for some $\alpha>0$. Equivalently, (M,d) is regular if the Hausdorff measure $\mathcal{H}_\alpha$ with $\alpha=\dim_H M$ is α-regular.
On a regular metric space, consider for any $\beta>0$ the quadratic form

$$\mathcal{E}_\beta(f,f)=\iint_{M\times M}\frac{(f(x)-f(y))^2}{d(x,y)^{\alpha+\beta}}d\mu(x)d\mu(y),$$

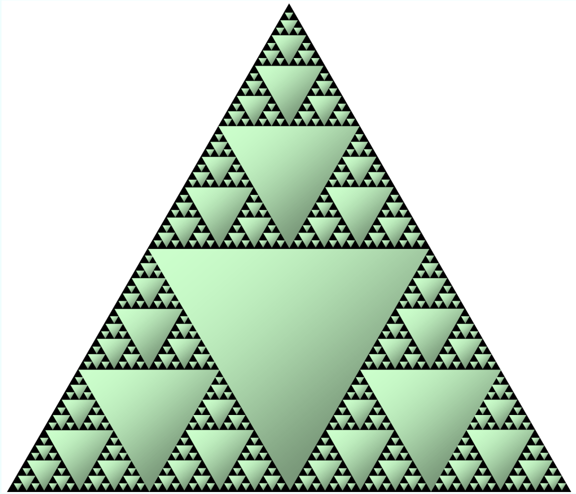
where $\mu=\mathcal{H}_\alpha$. Define the *walk dimension* β^* of (M,d) by

$$\beta^*=\sup\left\{\beta>0:\ \mathcal{E}_\beta \text{ is a regular Dirichlet form in } L^2(M,\mu)\right\}.$$

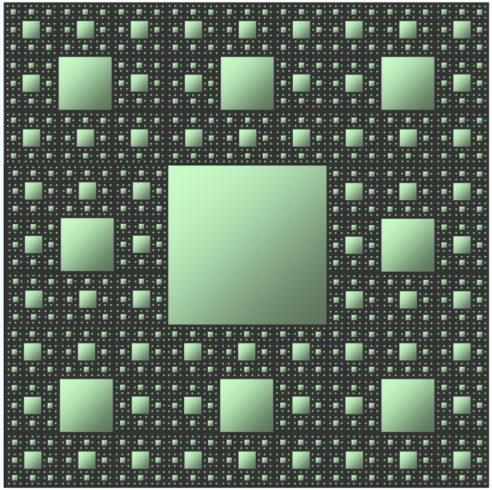
It is easy to show that always $\beta^*\geq 2$ (because $Lip_0(M)\subset \operatorname{dom}(\mathcal{E}_\beta)$).

For example:

- in $\mathbb{R}^n$ we have $\beta^*=2$;
- on ultra-metric spaces $\beta^*=\infty$ (by Theorem 3);
- on typical fractal spaces $2<\beta^*<\infty$.



Sierpinski gasket (SG), $\alpha = \frac{\log 3}{\log 2}$, $\beta^* = \frac{\log 5}{\log 3}$



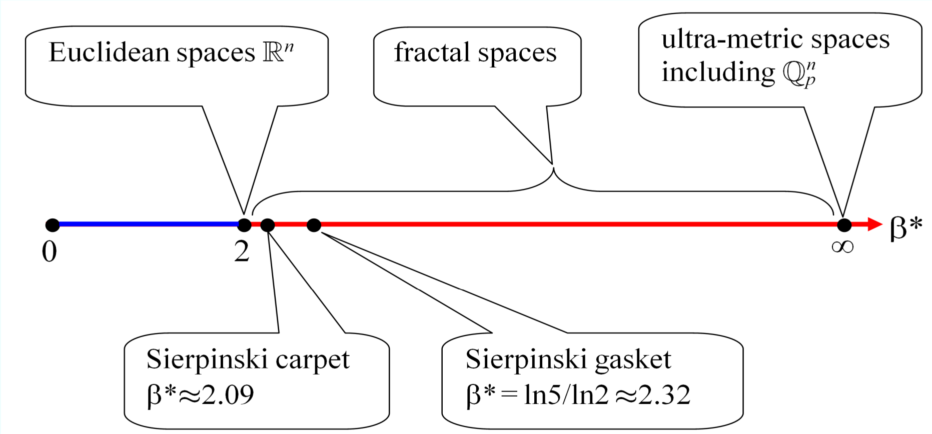
Sierpinski carpet (SC), $\alpha = \frac{\log 8}{\log 3}$, $\beta^* \approx 2.09$

On many fractal spaces including SG and SC, there exists a *local* Dirichlet form (and associated diffusion) whose heat kernel satisfies *sub-Gaussian* estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\gamma}} \exp\left(-c\left(\frac{d^\gamma(x,y)}{t}\right)^{\frac{1}{\gamma-1}}\right) \tag{18}$$

for some $\alpha > 0$ and $\gamma > 1$ (Barlow, Bass, Hambly, Kigami, Kumagai, Kusuoka, Perkins, et al.). If (18) is satisfied then the underlying metric measure space is necessarily α-regular and $\gamma = \beta^*$.

The walk dimension β^* is the second (after α) invariant of a regular metric space. Here any pair (α, β^*) with $\alpha > 0$ and $\beta^* \in [2, \infty]$ can be realized (M.Barlow).



Parameter α is responsible for *integration* on M as it determines measure, while β^* is responsible for *differentiation* on M as in many cases it determines the generator $\mathcal{L}$ of a local Dirichlet form (that is a natural Laplacian).

Open questions. Let (M, d) be a regular metric space.
1. Is it true that if $\beta^* = \infty$ then d is an ultra-metric?
2. Is it true that if $\beta^* < \infty$ then there exists a non-trivial *local* regular Dirichlet form on M and, hence, a diffusion on M?

Heat kernel estimates under relaxed hypotheses

Definition. We say that J satisfies the *β-Poincaré inequality* if, for any ball $B = B_r(x_0)$ and any function $f \in L^2(B, \mu)$,

$$\int_B \left|f - \overline{f}\right|^2 d\mu \le C r^\beta \iint_{B\times B} (f(x) - f(y))^2 J(x,y) d\mu(x) d\mu(y) \tag{PI}$$

where $\overline{f} = \fint_B f d\mu$ and the constant C is the same for all balls B and all f's.

Definition. We say that J satisfies the *β-tail condition* if, for any ball $B_r(x)$,

$$\int_{M\setminus B_r(x)} J(x,y) d\mu(y) \le C r^{-\beta}. \tag{TJ}$$

It is easy to verify that $J(x,y) \ge c d(x,y)^{-(\alpha+\beta)} \Rightarrow (PI)$ and

$$J(x,y) \le C d(x,y)^{-(\alpha+\beta)} \Rightarrow (TJ),$$

so that (PI) and (TJ) can be regarded as relaxed (integral) versions of the lower resp. upper bounds of $J(x,y)$.

In fact, both (PI) and (TJ) can be stated for *jump measures* $J(x, dy)$.

Theorem 4 *If (TJ) and (PI) are satisfied then the heat kernel $p_t(x,y)$ exists, is continuous in (t,x,y), Hölder continuous in (x,y) and satisfies the following weak upper bound*

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta} \tag{WUE}$$

and the near-diagonal lower bound

$$p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}}\mathbf{1}_{\left\{d(x,y)\le t^{1/\beta}\right\}}. \tag{NLE}$$

Moreover, under the standing assumption (TJ), we have

$$(PI) \Leftrightarrow (WUE)+(NLE).$$

Note that the exponent $-\beta$ in (WUE) does not match the exponent $-(\alpha+\beta)$ in the optimal heat kernel bound (15).

However, under the hypotheses (TJ) and (PI) alone, the estimates (WUE) and (NLE) cannot be essentially improved, as will be shown in examples below.

Sharpness of (WUE) and (NLE)

We use the heat kernel (17) on the product space M in order to show the sharpness of the estimates (WUE) and (NLE) under the hypotheses (PI) and (TJ).

It is easy to show that the jump measure J from (16) satisfies (TJ) and that $p_t(x, y)$ from (17) satisfies both (WUE) and (NLE) with parameters α and β. By extension of Theorem 4 to jump measures, the Poincaré inequality (PI) is also satisfied for J.

Consider the range of x, y such that

$$d_1(x_1, y_1) > t^{1/\beta} \text{ and } d_i(x_i, y_i) \le t^{1/\beta} \text{ for } i = 2, ..., n.$$

Then (17) yields

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d_1(x_1, y_1)}{t^{1/\beta}}\right)^{-(\alpha_1+\beta)} = \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha_1+\beta)}.$$

Since α_1 can be chosen arbitrarily small while keeping the same α, β, we see that (WUE) is optimal.

Similarly, consider the range of x, y such that

$$d_i(x_i, y_i) \simeq d_j(x_j, y_j) \quad \text{for all } i, j.$$

Then $d(x,y) \simeq d_i(x_i,y_i)$ and

$$\begin{aligned} p_t(x,y) &\simeq \frac{1}{t^{\alpha/\beta}} \prod_{i=1}^{n} \left(1 + \frac{d_i(x_i,y_i)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)} \\ &\simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+n\beta)} . \end{aligned}$$

Since n can be chosen arbitrarily large, while α and β are fixed, we see that no lower bound of the form

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-N}$$

can be guaranteed.

Semi-bounded kernels

In the above setting of an α-regular ultra-metric space (M, d, μ), consider separately upper and lower bounds of J :

$$J(x, y) \leq Cd(x, y)^{-(\alpha+\beta)} \tag{$J_\leq$}$$

$$J(x, y) \geq cd(x, y)^{-(\alpha+\beta)}. \tag{$J_\geq$}$$

Theorem 5 *If $(J_\leq)$ and (PI) are satisfied then the heat kernel satisfies for all $x, y \in M$ and $t > 0$ the optimal upper bound*

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{UE}$$

and the near-diagonal lower bound

$$p_t(x, y) \geq \frac{c}{t^{\alpha/\beta}} \quad \forall x, y \in M \quad \text{and } \forall t > d(x, y)^\beta. \tag{NLE}$$

In fact, we have

$$(J_\leq) + (PI) \Leftrightarrow (UE) + (NLE).$$

Theorem 6 *If* $(J_{\geq})$ *and* (TJ) *are satisfied then the heat kernel satisfies for all* $x, y \in M$ *and* $t > 0$ *the optimal lower bound*

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{LE}$$

and the weak upper bound

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}. \tag{WUE}$$

Moreover, under the standing assumption (TJ), *we have*

$$(J_{\geq}) \Leftrightarrow (WUE) + (LE).$$

Clearly, Theorems 5 and 6 imply that

$$(J) \Leftrightarrow (UE) + (LE),$$

which is equivalent to Theorem 3.